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The Path Integral Method to the Aharonov-Bohm Effect

Abstract

In this review of the Aharonov-Bohm effect, we look at the path integral method to solving the dynamics of the system. In particular, we examine the general form of the propagator and its relation to the multiply connectedness of the space. We determine the full propagator by summing over the partial propagators in each homotopy class. This result is compared with the solution to the Schrodinger equation as presented by Aharonov and Bohm themselves. We also evaluate the wave function for both cases and find that the predictions for the interference pattern match. The importance of the non-self-adjointness of the Hamiltonian is also made clear and an interpretation of the magnetic flux as an extension to the Hamiltonian is presented.

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Content

1 Introduction

1.1 The Aharonov-Bohm Effect

The laws that describe electromagnetism are usually expressed in terms of the electric field E and magnetic field B, but the Maxwell equations can be simplified by introducing the electric and magnetic potential. These potentials help to reduce the number of equations that need to be solved in a given problem; however, they were not at first considered anything more than a mathematical trick. In 1959, Aharonov and Bohm suggested that, in quantum mechanics, the electric scalar potential Φ and the magnetic vector potential A could have an influence on a charged particle and, therefore, have a certain physical reality to them (Aharonov and Bohm, Significance of Electromagnetic Potentials in the Quantum Theory). In quantum mechanics, the canonical formalism is needed to fully describe a system and the potentials can enter the propagator of a particle through the Hamiltonian or the Lagrangian, having a direct influence on a wave function's evolution. The Aharonov-Bohm effect has since been found experimentally, but the correct interpretation of the effect remains controversial (Chambers; Boyer).

In the magnetic version of the AB-effect, the authors describe the setup as follows. There is an infinitely long solenoid which produces a magnetic field. Because it is infinitely long, there are no field lines outside of the solenoid. There is, however, a magnetic vector potential everywhere in the space. More details on the setup can be found in the appendix. From a source, electrons are sent to a detector on the other side of the solenoid. Some electrons will pass the solenoid from the left and others from the right. Even though the electrons never enter a field because they are prevented from entering the solenoid by a potential barrier, and therefore never experience any force, there will be a clear effect on the phase of the electron's wave function. To see how this comes about, consider the Lagrangian for an electromagnetic setup.

$$L(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \frac{1}{2} m \dot{\vec{r}}^2 + \frac{q}{c} \dot{\vec{r}} \cdot \vec{A} - q\Phi \tag{1.01}$$

The magnetic vector potential, therefore, enters the propagator.

$$\exp\left(\frac{i}{\hbar} S(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1)\right) = \exp\left(\frac{i}{\hbar} x + \frac{iq}{\hbar c} \int_{t_1}^{t_2} dt\, \dot{\vec{r}} \cdot \vec{A}\right) = \exp\left(\frac{i}{\hbar} x + \frac{iq}{\hbar c} \int_{\overrightarrow{r_1}}^{\overrightarrow{r_2}} d\vec{r}\, \vec{A}\right) \tag{1.02}$$

$$\text{where } x = \int_{t_1}^{t_2} dt\, \frac{1}{2} m \dot{\vec{r}}^2 - q\Phi \tag{1.03}$$

On the detector, the paths going left and right of the solenoid are combined. The difference in phase, when the two electrons interfere, is the usual phase difference plus a term contributed by the vector potential A. Using Stokes's theorem, we can show that the size of this extra term, α, is related to the strength of the magnetic flux, F.

$$\frac{q}{\hbar c}\left(\int_{\overrightarrow{r_1}}^{\overrightarrow{r_2}} d\overrightarrow{r_a}\, \vec{A} - \int_{\overrightarrow{r_1}}^{\overrightarrow{r_2}} d\overrightarrow{r_b}\, \vec{A}\right) = \frac{q}{\hbar c} \oint_R d\vec{l}\, \vec{A} = \frac{q}{\hbar c} \int_S ds\, \vec{\nabla} \times \vec{A} = \frac{q}{\hbar c} \int_S ds\, \vec{B} = \frac{q}{\hbar c} F = \alpha \tag{1.04}$$

Therefore, the strength of the magnetic field has a direct influence on a part of the system where there is no magnetic field. This also shows that when the magnetic flux running through the solenoid, multiplied by $q/(\hbar c)$, is a multiple of 2π, the AB-effect is not noticeable and the original interference pattern can be observed.

1.2 The Berry Phase

The AB-effect has been generalised by Berry as a particular type of geometric phase. A geometric phase can arise when a quantum system is transported around some circuit by varying parameters in the Hamiltonian. From the adiabatic theorem, it follows that the system will remain in the same state when this transport is done slowly. When the Hamiltonian is finally returned to its original form, the system will be in the same state apart from an additional phase factor (Berry, Quantal Phase Factors accompanying Adiabatic Changes). The cause for this anholonomy is the fact that the circuit lies on a curved surface in parameter space. The phase factor does not depend on the energy of the system or on the time it takes to circle the loop; it is only affected by the geometry of the space (Berry, The Geometric Phase). In the case of the AB-effect, the geometry is determined by the magnetic flux and, like we just did above, the phase can be calculated by integrating over any surface that catches all the flux through the circuit.

1.3 Multivaluedness and Self-Adjointness

The AB-setup which we are considering does not involve a simply connected space due to the presence of the magnetic flux. The space is multiply connected with a puncture at the position of the solenoid. In such a space, the Hamiltonian would normally lead to a multivalued wave function. As solutions to the Schrodinger equation, such functions are, in general, not permissible in quantum mechanics and it is easy to reason why they would not be appropriate in this case either. In principle, the magnetic flux could always be decreased adiabatically to zero, turning the space back to a simply connected region. If multivalued wave functions were allowed for the AB-setup, these functions would also need to be allowed for a free particle. Quantum mechanics prohibits the use of multivalued wave functions in ordinary space and, therefore, its use is also prohibited in the AB-setup (Aharonov and Bohm, Further Considerations on Electromagnetic Potentials in the Quantum Theory). The fact that the particle was in a multiply connected region at some stage in its history, before the flux was turned off adiabatically, is of no importance as systems in quantum mechanics are independent of the past history.

Besides the fact that the space is multiply connected, there is a requirement that the particle cannot penetrate the solenoid. This is simply a boundary condition which states that the wave function is zero at the origin. The problems with the wave function which we mentioned stem from the fact that the Hamiltonian in this case is not self-adjoint, which means that it does not adequately describe the dynamics of the system (Thaller). According to Stone's theorem, the self-adjointness of the Hamiltonian operator is equivalent to the existence of a quantum-mechanical unitary time evolution; it is the generator of is a unique one-parameter unitary group (Le Bellac). The multiply connectedness of the setup results in an ambiguous time evolution. An extra piece of information is needed to be able to describe the system fully; this information lies with the boundary conditions (Schulman, Approximate Topologies). We will end up with a one-parameter family of extensions of the Hamiltonian. Basically, the original definition of the Hamiltonian was not appropriate for this system and a new operator needs to be found which we can then define to be the Hamiltonian (Garbaczewski and Karwowski; Carlen and Loffredo). This is done in the appendix and is used in solving the Schrodinger equation.

2 The Schrodinger Equation

2.1 The Exact Solution

In the original paper, Aharonov and Bohm gave a solution for the wave function of the magnetic version (Aharonov and Bohm, Significance of Electromagnetic Potentials in the Quantum Theory). By means of rewriting the Schrodinger equation, which has been redone in the appendix, the authors got the following expression.

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi}+\mathrm{i}\alpha\right)^2+\mathrm{k}^2\right)\psi=0 \tag{2.01}$$

It follows that the general solution includes Bessel functions and will have the form:

$$\psi=\sum_{\mathrm{m}=-\infty}^{\infty}\exp(\mathrm{i}\mathrm{m}\varphi)\left(\mathrm{a_m}\mathrm{J}_{\mathrm{m}+\alpha}(\mathrm{k}\rho)+\mathrm{b_m}\mathrm{J}_{-\mathrm{m}-\alpha}(\mathrm{k}\rho)\right) \tag{2.02}$$

The coefficients a(m) and b(m), here, need to be chosen so that the wave function satisfies the boundary conditions of the AB-setup. The expression can then be simplified to find a comprehensive solution for the wave function. A discussion of this process is given in the original paper where the authors verify that these coefficients must be:

$$\mathrm{a_m}=(-\mathrm{i})^{|\mathrm{m}+\alpha|} \tag{2.03}$$

$$\mathrm{b_m}=0 \tag{2.04}$$

The final solution for the wave function, with k = 1, includes two terms, where the first represents the incident wave and the second term corresponds to the scattering wave.

$$\psi=\exp(-\mathrm{i}\alpha\varphi-\mathrm{i}\rho\cos(\varphi))+\frac{\exp(\mathrm{i}\rho)}{(2\pi\mathrm{i}\rho)^{\frac{1}{2}}}\sin(\pi\alpha)\frac{\exp\left(-\mathrm{i}\frac{\varphi}{2}\right)}{\cos\left(\frac{\varphi}{2}\right)} \tag{2.05}$$

2.2 Non-Self-Adjoint Hamiltonian

To stress the point of the non-self-adjointness of the Hamiltonian, we will make an analogy with the system of the particle in a box. When one naively uses the original definition of the Hamiltonian without taking into account the influence of the boundary conditions on the dynamics, some problems may arise (Bonneau, Faraut and Valent). For a particle in a box of width L, the Hamiltonian and its domain can be defined as:

$$\mathrm{H}=-\frac{\hbar^2}{2\mathrm{m}}\vec{\nabla}^2 \tag{2.06}$$

$$\text{for } \mathrm{D(H)}=\left\{\psi,\mathrm{H}\psi\in\mathrm{L}^2\left(-\frac{\mathrm{L}}{2},\frac{\mathrm{L}}{2}\right),\psi\left(\pm\frac{\mathrm{L}}{2}\right)=0\right\} \tag{2.07}$$

We may consider an arbitrary function which satisfies the boundary conditions and is zero outside the box.

$$\psi=-\left(\frac{30}{\mathrm{L}^5}\right)^{\frac{1}{2}}\left(\mathrm{x}^2-\left(\frac{\mathrm{L}}{2}\right)^2\right) \tag{2.08}$$

$$H\psi = \overline{\psi} = \frac{\hbar^2}{m}\left(\frac{30}{L^5}\right)^{\frac{1}{2}} \tag{2.09}$$

The action of the Hamiltonian on the wave function returns a constant. In this case, it can be shown that the use of the self-adjointness of the operator produces a paradox.

$$\langle E^2 \rangle = \langle H\psi | H\psi \rangle = \langle \overline{\psi} | \overline{\psi} \rangle = \frac{30\hbar^4}{m^2 L^4} \tag{2.10}$$

$$\langle E^2 \rangle = \langle \psi | H^2 \psi \rangle = \langle \psi | H\overline{\psi} \rangle = 0 \tag{2.11}$$

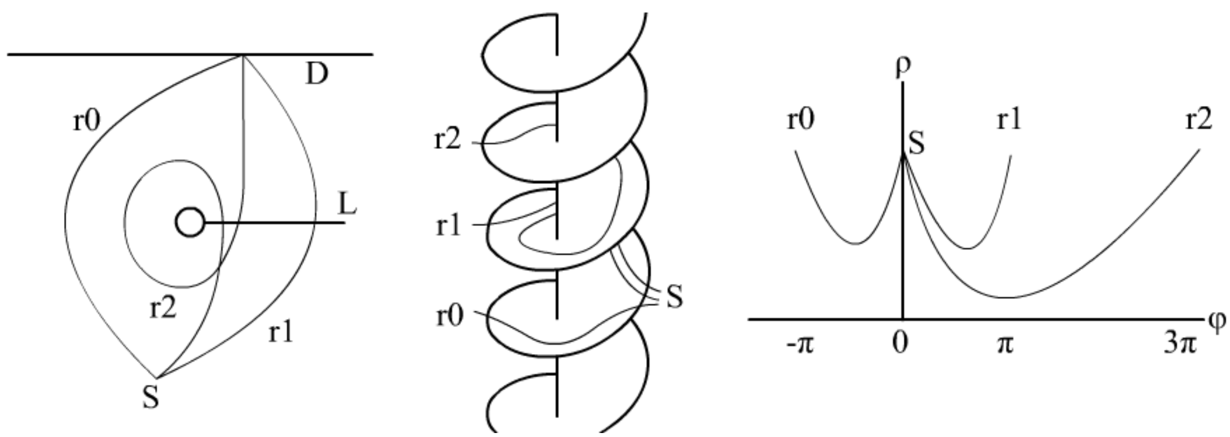
This contradiction arises because the definition of the Hamiltonian is not appropriate; the operator is not self-adjoint. The system for the AB-effect has a similar problem due to the multiply connectedness of the space. The boundary condition that the wave function must vanish within the solenoid causes an ambiguity in the dynamics of the system. The Hamiltonian that does define the dynamics well has, in the appendix, been shown to be:

$$H_\alpha = \frac{\hbar^2}{2m}\left(-\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\left(i\frac{\partial}{\partial\varphi} - \alpha\right)^2\right) \tag{2.12}$$

3 The Path Integral

3.1 The Covering Space

In order to deal with the multivaluedness and with the non-self-adjointness of the Hamiltonian, we can move from the base space to a covering space (Schulman, Approximate Topologies). In this simply connected space, a position with angle $\varphi = \pi \neq 3\pi$ and the Hamiltonian is once again self-adjoint. When the solenoid is turned on, the covering space can be thought of as the Riemann surface for the logarithm as seen in figure 1.



In the covering space, a path going to the left of the solenoid is differentiated from one going to the right, i.e. it hits the detector at a different location with a different angle φ. The accumulated phase factor for these paths, therefore, is also different. However, when the distance between each level of the Riemann surface is exactly 2π, no difference between paths can be determined. This corresponds to the case when the phase factor α, proportional to the magnetic flux running through the solenoid, is a multiple of 2π and the AB-effect is not noticeable, as explained above. The Riemann surface can be stretched and flattened out to get a simple simply connected two dimensional half-space with radial direction ρ and angle φ running from -∞ to ∞. In the base space, it is easy to identify the number of windings n around the solenoid the path makes. This number can be determined by looking at the amount of times a particle crosses the line L from the bottom up, subtracting the numbers of times it crosses this line from the top down. In the covering space, a path with winding number n originates from the source S located at position $\rho = \rho(1)$, $\varphi = \varphi(1) = 0$ and can travel to the detector D at position $\rho = \rho(2)$, $\varphi = \varphi(2) + 2\pi n$.

3.2 The General Form

In a multiply connected space, there are many distinct homotopy classes of paths which, by definition, cannot be continuously deformed into each other. Every homotopy class itself is located on a simply connected region of the base space of which the dynamics are determined by the propagator for that particular homotopy class. The full propagator for the multiply connected space is, then, the sum over all partial propagators for the different classes (Schulman, A Path Integral for Spin). For the AB-setup, the space is multiply connected due to the presence of the solenoid; homotopy classes are classified by the number of windings around the solenoid. There is one additional degree of freedom that such a system presents which stems from the fact that there is no prior reason for every class to have the same weight. It turns out that this weight is simply a phase factor dependent on the homotopy class (Dowker; Laidlaw and Morette-DeWitt). The general form of the propagator, therefore, is:

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \sum_{n=-\infty}^{\infty} \exp(in\alpha)\, K_n(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) \tag{3.01}$$

To see why it is necessary for there to be such an extra phase factor, let us look at what would happen when we take the end point r(2) and move it all the way around the puncture in the space in a anticlockwise manner and return to the same position we started from. For the full propagator, nothing changed physically and so the difference can only be a phase factor which we may call -α, where the minus is added for consistency later on. On the other hand, each individual partial propagator is equivalent to the original partial propagator with n + 1.

$$\exp(-i\alpha)\, K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \sum_{n=-\infty}^{\infty} A_n K_{n+1}(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \sum_{n=-\infty}^{\infty} A_{n-1} K_n(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) \tag{3.02}$$

The change of variables from n→ n - 1 is allowed because the sum runs from -∞ to ∞. We can see, by comparing the original propagator with this one, that a next iteration of A(n) adds only a phase factor of size α.

$$A_{n+1} = \exp(i\alpha)\, A_n \tag{3.03}$$

The choice for A(0) is arbitrary and can be put to 1. This will results in a very simple expression for A(n).

$$A_n = A_0 \exp(in\alpha) = \exp(in\alpha) \tag{3.04}$$

It is, therefore, obvious that the full propagator for a multiply connected space is the sum over all partial propagators with an additional phase factor dependent on the homotopy class (Schulman, Techniques And Applications Of Path Integration). Likewise, it becomes obvious that the partial propagators are not independent of each other. By looking at figure 1 we deduce that:

$$K_n(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = K_0(\overrightarrow{r_2} + 2\pi n\hat{\varphi}, t_2, \overrightarrow{r_1}, t_1) \tag{3.05}$$

Therefore, the traverse of an extra 2π through the covering space adds a phase α to the propagator. The origin and interpretation of this α must now be found. In the case of the AB-effect, the magnetic flux provides such an origin as will be obvious later, when the full propagator for this system is found. As for the interpretation, the magnetic vector potential can be said to directly influence the phase factor, but, as we will see later on, α is also closely connected to the self-adjointness of the Hamiltonian. Let us first look at what will happen to the covering space when the solenoid is turned on. A magnetic vector potential arises which points only in the φ-direction and has a strength which falls off with $1/\rho$. This potential, which circles round the solenoid, is the origin of the multivaluedness of the system, as depicted by the Riemann surface. In the stretched and flattened version of the covering space, however, the description of the magnetic vector potential becomes very simple with all vectors of same magnitude, pointing in the same direction. It may be visualised as the slope of a straight, downward sloping hill, i.e. it is the gradient of a smooth scalar potential Ω. In the covering space, the complicated magnetic vector potential becomes a simple gauge transformation. We will use this very important fact to find the propagator for the AB-effect.

3.3 The Propagator

In this section, we will try to find the propagator for the AB-effect in an intuitive manner by combining the works of Schulman and of Gerry and Singh (Schulman, Approximate Topologies; Gerry and Singh). Let us start by considering a space described in cylindrical coordinates, with an infinitely long solenoid along the z-axis. The width of the solenoid is limited to 0 and, for the time being, the solenoid is turned off. The system is completely independent of z and can be reduced to two dimensions. We are, therefore, considering a punctured, multiply connected space with an infinitesimally small hole at the origin. The propagator is given by:

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \int D\vec{r}\ \exp\left(\frac{i}{\hbar} S(\overrightarrow{r_2}, \overrightarrow{r_1})\right) = \int D\vec{r}\ \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, L\left(\dot{\vec{r}}, \vec{r}\right)\right) \tag{3.06}$$

$$\text{where } S(\overrightarrow{r_2}, \overrightarrow{r_1}) = \int_{t_1}^{t_2} dt\, L\left(\dot{\vec{r}}, \vec{r}\right) \tag{3.07}$$

The variable of integration Dr denotes that the integral is taken over all possible paths between r(1) and r(2) in the time interval t(1) to t(2). The propagator can be rewritten to include an angular component; however, nothing changes to the propagator as the integration over the delta-function cancels itself out. We are now considering the propagator in the covering space in which the angle φ does not run from $-\pi$ to π but from $-\infty$ to ∞. The delta function selects out those paths which have a specific angle φ. As explained above, every time the φ runs passed $\pi/2 + 2\pi n$, i.e. passed the line L in figure 1, it becomes part of a higher homotopy class. The delta function, therefore, can be associated with the summing over all different homotopy classes, as will become clearer near the end.

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \int D\vec{r}\ \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} dt\, L\left(\dot{\vec{r}}, \vec{r}\right)\right)\int_{-\infty}^{\infty} d\theta\, \delta\left(\theta - \int_{t_1}^{t_2} dt\, \dot{\varphi}\right) \tag{3.08}$$

From Fourier analysis we know that the delta function is the inverse Fourier transform of the number 1, which we can, subsequently, use in our propagator.

$$2\pi\delta(x) = \int_{-\infty}^{\infty} d\lambda \ \exp(i\lambda x) \tag{3.09}$$

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \int D\vec{r} \ \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} dt \, L(\dot{\vec{r}}, \vec{r})\right) \int_{-\infty}^{\infty} d\theta \ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \ \exp\left(i\lambda\left(\theta - \int_{t_1}^{t_2} dt \, \dot{\varphi}\right)\right) \tag{3.10}$$

By rearranging the terms and combining the integral over the Lagrangian with the integral over the angular component, we obtain:

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \int_{-\infty}^{\infty} d\theta \ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \ \exp(i\lambda\theta) \int D\vec{r} \ \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} dt \, L(\dot{\vec{r}}, \vec{r}) - \lambda\hbar\dot{\varphi}\right) \tag{3.11}$$

The integrand of the integral over λ, excluding the phase factor in front and defined as K(λ), becomes:

$$K_\lambda(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \int D\vec{r} \ \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} dt \, L(\dot{\vec{r}}, \vec{r}) - \lambda\hbar\dot{\varphi}\right) = \int D\vec{r} \ \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} dt \ \frac{1}{2} m\dot{\vec{r}}^2 - \lambda\hbar\dot{\varphi}\right) \tag{3.12}$$

$$\text{where } L(\dot{\vec{r}}, \vec{r}) = \frac{1}{2} m\dot{\vec{r}}^2 \tag{3.13}$$

The Lagrangian is that for a free particle as we are still looking at a system with the solenoid turned off, i.e. no magnetic field or potential. Here, we could have chosen an alternative method of finding the propagator by including the magnetic vector potential within the Lagrangian, as was done by Gerry and Singh, however, this method is less intuitive than the one we present here. The integrand of the expression above can be discretised in the usual manner.

$$K_\lambda(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \lim_{N\to\infty} A_N \int \prod_{j}^{N-1} d\overrightarrow{r_j} \ \exp\left(\frac{i}{\hbar}\sum_{j=1}^{N} S_l(\overrightarrow{r_2}, \overrightarrow{r_1})\right) \tag{3.14}$$

The action is broken into small parts where each step is the area underneath the Lagrangian given by:

$$\lim_{N\to\infty} S_l(\overrightarrow{r_j}, \overrightarrow{r_{j-1}}) = \frac{m}{2\varepsilon}\left(\rho_j{}^2 + \rho_{j-1}{}^2\right) + \frac{m}{\varepsilon}\rho_j\rho_{j-1}\cos\left(\varphi_j - \varphi_{j-1}\right) - \lambda\hbar\left(\varphi_j - \varphi_{j-1}\right) \tag{3.15}$$

$$\text{where } \varepsilon = t_j - t_{j-1} \tag{3.16}$$

By determining the Taylor expansion of the cosine, it can be shown that:

$$\cos\left(\Delta\varphi_j\right) - x\varepsilon\Delta\varphi_j = \cos\left(\Delta\varphi_j + x\varepsilon\right) + \frac{1}{2}x^2\varepsilon^2 \tag{3.17}$$

$$\text{where } \Delta\varphi_j = \varphi_j - \varphi_{j-1} \tag{3.18}$$

Doubts can be raised on the integrity of this trick as it remains uncertain what the effect would be of higher order interactions on the path integral. Ignoring this for the moment, it can be applied to the action to obtain:

$$S_l(\overrightarrow{r_j}, \overrightarrow{r_{j-1}}) = \frac{m}{2\varepsilon}\left(\rho_j{}^2 + \rho_{j-1}{}^2\right) + \frac{m}{\varepsilon}\rho_j\rho_{j-1}\left(\cos\left(\Delta\varphi_j + \frac{\varepsilon}{m\rho_j\rho_{j-1}}\lambda\hbar\right) + \frac{1}{2}\left(\frac{\varepsilon}{m\rho_j\rho_{j-1}}\lambda\hbar\right)^2\right) \tag{3.19}$$

The integrand of K(λ) includes the exponential of the sum over all small partial-actions, which can be rewritten as the product of the exponentials of the actions.

$$\exp\left(\frac{i}{\hbar}\sum_{j=1}^{N} S_l(\overrightarrow{r_j}, \overrightarrow{r_{j-1}})\right) = \prod_{j=1}^{N} \exp\left(\frac{i}{\hbar} S_l(\overrightarrow{r_j}, \overrightarrow{r_{j-1}})\right) \tag{3.20}$$

This, in turn, allows us to consider the exponential of the actions without worrying about the summation.

$$\exp\left(\frac{im}{2\hbar\varepsilon}\left(\rho_j{}^2+\rho_{j-1}{}^2\right)+\frac{im}{\hbar\varepsilon}\rho_j\rho_{j-1}\cos\left(\Delta\varphi_j+\frac{\varepsilon}{m\rho_j\rho_{j-1}}\lambda\hbar\right)+\frac{im}{2\hbar\varepsilon}\rho_j\rho_{j-1}\left(\frac{\varepsilon}{m\rho_j\rho_{j-1}}\lambda\hbar\right)^2\right) \tag{3.21}$$

It can be shown that the exponential of a cosine is equal to a sum over modified Bessel functions.

$$\exp(y\cos(x))=\sum_{\nu=-\infty}^{\infty}\exp(i\nu x)\,I_\nu(y) \tag{3.22}$$

The exponential of the actions can, then, be rewritten to obtain:

$$\sum_{\nu=-\infty}^{\infty}\exp\left(i\nu\left(\Delta\varphi_j+\frac{\varepsilon}{m\rho_j\rho_{j-1}}\lambda\hbar\right)\right)I_\nu\left(\frac{im}{\hbar\varepsilon}\rho_j\rho_{j-1}\right)\exp\left(\frac{im}{2\hbar\varepsilon}\left(\rho_j{}^2+\rho_{j-1}{}^2\right)+i\frac{\varepsilon\hbar\lambda^2}{2m\rho_j\rho_{j-1}}\right) \tag{3.23}$$

Rearranging, we arrive at:

$$\sum_{\nu=-\infty}^{\infty}\exp\left(i\nu\Delta\varphi_j\right)\exp\left(\frac{im}{2\hbar\varepsilon}\left(\rho_j{}^2+\rho_{j-1}{}^2\right)\right)I_\nu\left(\frac{im}{\hbar\varepsilon}\rho_j\rho_{j-1}\right)\exp\left(i\nu\frac{\varepsilon}{m\rho_j\rho_{j-1}}\lambda\hbar+i\frac{\varepsilon\hbar\lambda^2}{2m\rho_j\rho_{j-1}}\right) \tag{3.24}$$

It follows from the asymptotic behaviour of the modified Bessel functions that the following approximation holds when the argument of x is small (Inomata and Singh).

$$I_{\nu+y}(x)\sim\left(\frac{1}{2\pi x}\right)^2\exp\left(x-\frac{1}{2x}\left((\nu+y)^2-\frac{1}{4}\right)\right) \tag{3.25}$$

$$\text{for }-\frac{1}{2}\pi\leq\arg(x)\leq\frac{1}{2}\pi \tag{3.26}$$

The exponential part written after the modified Bessel function can, therefore, be taken in by this function as a shift in the argument ν.

$$\sum_{\nu=-\infty}^{\infty}\exp\left(i\nu\Delta\varphi_j\right)\exp\left(\frac{im}{2\hbar\varepsilon}\left(\rho_j{}^2+\rho_{j-1}{}^2\right)\right)I_{\nu-\lambda}\left(\frac{-im}{\hbar\varepsilon}\rho_j\rho_{j-1}\right) \tag{3.27}$$

The part of the propagator we defined K(λ) now becomes:

$$\lim_{N\to\infty}A_N\int\prod_j^{N-1}d\vec{r_j}\prod_{j=1}^{N}\sum_{\nu=-\infty}^{\infty}\exp\left(i\nu\Delta\varphi_j\right)\exp\left(\frac{im}{2\hbar\varepsilon}\left(\rho_j{}^2+\rho_{j-1}{}^2\right)\right)I_{\nu-\lambda}\left(\frac{-im}{\hbar\varepsilon}\rho_j\rho_{j-1}\right) \tag{3.28}$$

We can extract the angular part of this propagator by making use of the orthogonality relations of the Kronecker delta function (Inomata and Singh).

$$\int_0^{2\pi}d\varphi\;\exp(i(x-y)\varphi)=2\pi\delta(x,y) \tag{3.29}$$

$$\sum_{\nu=-\infty}^{\infty}\exp\left(i\nu(\varphi_2-\varphi_1)\right)\lim_{N\to\infty}2\pi^{N-1}A_N\int\prod_j^{N-1}d\rho_j\prod_{j=1}^{N}\exp\left(\frac{im}{2\hbar\varepsilon}\left(\rho_j{}^2+\rho_{j-1}{}^2\right)\right)I_{\nu+\lambda}\left(\frac{-im}{\hbar\varepsilon}\rho_j\rho_{j-1}\right) \tag{3.30}$$

The full propagator can now be written as:

$$K(r_2,t_2,r_1,t_1)=\int_{-\infty}^{\infty}d\theta\;\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\;\exp(i\lambda\theta)\sum_{\nu=-\infty}^{\infty}\exp\left(i\nu(\varphi_2-\varphi_1)\right)Q_{\nu+\lambda} \tag{3.31}$$

$$\text{where }Q_{\nu+\lambda}=\lim_{N\to\infty}(2\pi)^{N-1}A_N\int\prod_j^{N-1}d\rho_j\prod_{j=1}^{N}\exp\left(\frac{im}{2\hbar\varepsilon}\left(\rho_j{}^2+\rho_{j-1}{}^2\right)\right)I_{\nu+\lambda}\left(\frac{-im}{\hbar\varepsilon}\rho_j\rho_{j-1}\right) \tag{3.32}$$

By making the change of variables $\lambda \rightarrow \lambda - \nu$, we obtain:

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \int_{-\infty}^{\infty} d\theta \, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\nu=-\infty}^{\infty} \exp(i\nu(\varphi_2 - \varphi_1 - \theta) + i\lambda\theta) \, Q_\lambda \tag{3.33}$$

We can use the Poisson summation formula to reinstate the delta function (Bernido and Inomata).

$$\sum_{\nu=-\infty}^{\infty} \exp(i\nu\varphi) = \sum_{n=-\infty}^{\infty} 2\pi \, \delta(\varphi + 2\pi n) \tag{3.34}$$

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\lambda \sum_{n=-\infty}^{\infty} \delta(\varphi_2 - \varphi_1 - \theta + 2\pi n) \exp(i\lambda\theta) \, Q_\lambda \tag{3.35}$$

The integral over the delta function can be changed into a discrete sum over n. Here, we identify $2\pi n$ as the extra angle φ a path would get by winding once more around the punctured origin. The propagator is, therefore, a sum over all homotopy classes with winding number n.

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \sum_{n=-\infty}^{\infty} K_n(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda \, \exp\big(i\lambda(\varphi_2 - \varphi_1 + 2\pi n)\big) \, Q_\lambda \tag{3.36}$$

The discretised integral $Q(\lambda)$ over all $\rho(j)$ has been evaluated by Peak and Inomata and, after intensive manipulation, reads as follows (Peak and Inomata).

$$Q_\lambda = \frac{-im}{2\pi\hbar(t_2 - t_1)} \exp\left(\frac{im}{2\hbar(t_2 - t_1)} ({\rho_1}^2 + {\rho_2}^2)\right) I_\lambda\left(\frac{-im\rho_1\rho_2}{\hbar(t_2 - t_1)}\right) \tag{3.37}$$

$$\text{with } A_N = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^N \tag{3.38}$$

We finally find the full expression for the propagator in a potential free environment.

$$\frac{-im}{2\pi\hbar\Delta t} \exp\left(\frac{im}{2\hbar\Delta t} ({\rho_1}^2 + {\rho_2}^2)\right) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda \, \exp\big(i\lambda(\varphi_2 - \varphi_1 + 2\pi n)\big) \, I_\lambda\left(\frac{-im\rho_1\rho_2}{\hbar\Delta t}\right) \tag{3.39}$$

$$\text{where } \Delta t = t_2 - t_1 \tag{3.40}$$

As was explained above, the addition of a magnetic potential produced by a solenoid is non-trivial in the base space, but is a simple gauge transformation is the covering space. The circular vector potential becomes a gradient of a scalar function Ω where the difference between the values for this Ω is equal to the line integral over the magnetic vector potential A between the two positions (Schulman, Approximate Topologies).

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) \rightarrow K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) \exp\left(\frac{iq}{\hbar c}\big(\Omega(\overrightarrow{r_2}) - \Omega(\overrightarrow{r_1})\big)\right) \tag{3.41}$$

When we return to the base space, we see that this scalar potential increases by the magnitude of the circuit integral over the magnetic vector potential every 2π, i.e. the difference in Ω over 2π is equal to the magnetic flux of the solenoid.

$$\Omega(\overrightarrow{r_{n+1}}) = \Omega(\overrightarrow{r_n}) + \oint_R d\vec{l}\,\vec{A} \tag{3.42}$$

The initial value for this scalar potential is arbitrary as it is simply a phase factor in front of the propagator and can, therefore, be taken to be 0.

$$\Omega(\overrightarrow{r_n}) = \Omega(\overrightarrow{r_0}) + 2\pi n A_0 = nF \tag{3.43}$$

$$\text{where } \Omega(\overrightarrow{r_0}) = 0 \tag{3.44}$$

$$\text{and } F = 2\pi A_0 \tag{3.45}$$

The propagator, which already was a sum over all homotopy classes, can now be rewritten to include a phase factor dependent on the scalar potential and the number of windings of that class. The second term in the exponential indicates what fraction of the magnetic vector potential is has passed through before hitting the detector.

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \sum_{n=-\infty}^{\infty} K_n \exp\left(\frac{iq}{\hbar c} nF + \frac{iq}{\hbar c} F \frac{(\varphi_2 - \varphi_1)}{2\pi}\right) \tag{3.46}$$

Because at any position on the detector the interference depends only on the relative phase, this second term can be removed. We finally see that, as mentioned above, the propagator is a combination of partial propagators with a weight equal to a phase factor α, determined by the magnetic flux.

$$K(\overrightarrow{r_2}, t_2, \overrightarrow{r_1}, t_1) = \sum_{n=-\infty}^{\infty} K_n \exp\left(\frac{iq}{\hbar c} nF\right) = \sum_{n=-\infty}^{\infty} K_n \exp(in\alpha) \tag{3.47}$$

Following our discussion of the non-self-adjointness of the Hamiltonian, we can give another interpretation of the parameter α. It was argued that there was need for a family of extension of the Hamiltonian by making use of an extra piece of information. For every partial propagator, the original Hamiltonian is, therefore, extended using the information hidden in the boundary condition, i.e. using the parameter α multiplied by the number of times the paths of this propagator loop around the solenoid. The full propagator is, then, the sum over all partial propagators for the different homotopy classes with the appropriate phase factor as the extension to the Hamiltonian of that class.

4 The Interference Pattern

4.1 The Schrodinger Equation Method

The solution to the AB-effect as presented by Aharonov and Bohm themselves causes a shift in the interference pattern seen on a detector. This effect can be made obvious by looking at the cross section and scattering amplitude of the wave function. The number of particles that scatter from the solenoid into the infinitesimal angle dφ is the number of incident particles multiplied by the infinitesimal ratio dσ for that angle.

$$dN_s = N_i \frac{d}{d\varphi} \sigma d\varphi \tag{4.01}$$

Here, dσ/dφ is called the differential cross section. By integrating over φ we could obtain the total cross section and total number of scattered particles. In order to determine this cross section, we first write the wave function as:

$$\psi = \psi_i + \psi_s = \exp(-i\rho\cos(\varphi) - i\alpha\varphi) + f(\varphi)\frac{\exp(i\rho)}{(\rho)^{\frac{1}{2}}} \tag{4.02}$$

$$\text{where } f(\varphi) = \frac{1}{(2\pi i)^{\frac{1}{2}}} \sin(\pi\alpha) \frac{\exp\left(-i\frac{\varphi}{2}\right)}{\cos\left(\frac{\varphi}{2}\right)} \tag{4.03}$$

It is shown in the appendix that there is a simple relation between what is called the scattering amplitude f(φ) and the differential cross section dσ/dφ.

$$\frac{d}{d\varphi}\sigma = |f(\varphi)|^2 \tag{4.04}$$

In order to find the expression we are looking for, we simply take the modulus squared of the scattering amplitude.

$$\left|\frac{1}{(2\pi i)^{\frac{1}{2}}}\sin(\pi\alpha)\frac{\exp\left(-i\frac{\varphi}{2}\right)}{\cos\left(\frac{\varphi}{2}\right)}\right|^2 = \frac{\sin^2(\pi\alpha)}{2\pi\cos^2\left(\frac{\varphi}{2}\right)} \tag{4.05}$$

We see that, indeed, there is a periodic dependence on α and that, due to the square of the sine function, the addition of an integer to α gives the same interference pattern.

4.2 The Path Integral Method

The path integral method has produced a propagator for the AB-effect which can be used to determine the full wave function for this system. We will try to find an expression for the interference pattern, i.e. the differential cross section, using this method and compare it with the previous solution. First, we must find the wave function using the propagator, this was done by Sakoda and Omote using the Lippmann–Schwinger equation; we simply state their result (Sakoda and Omote).

$$\psi = \psi_i + \psi_s = \sum_{m=-\infty}^{\infty} J_{|m+\alpha|}(k\rho)\exp\left(im\varphi - |m+\alpha|\frac{\pi}{2}\right) \tag{4.06}$$

$$\text{with } \psi_i = \exp(-ik\rho\cos(\varphi)) \tag{4.07}$$

$$\text{and } \psi_s = -\exp(-ik\rho\cos(\varphi)) + \sum_{m=-\infty}^{\infty} J_{|m+\alpha|}(k\rho)\exp\left(im\varphi - |m+\alpha|\frac{\pi}{2}\right) \tag{4.08}$$

We see that, even though these authors have used a standard plane wave as the incident wave, different from Aharonov and Bohm, the final result for the wave function is very similar. However, the scattering part separately is distinct and might give a different differential cross section. When observing at $\rho \to \infty$, the scattering wave can be written as:

$$\psi_s = -\frac{1}{(2\pi k\rho)^{\frac{1}{2}}}\exp\left(ik\rho - i\frac{\pi}{4}\right)\sum_{m=-\infty}^{\infty}(\exp(-\operatorname{sign}(m+\alpha)\, i\pi\alpha) - 1)\exp(im\varphi) \tag{4.09}$$

When put into the usual form, we see that f(φ) becomes:

$$f(\varphi) = -\frac{1}{(2\pi k)^{\frac{1}{2}}}\sum_{m=-\infty}^{\infty}(\exp(-\operatorname{sign}(m+\alpha)\, i\pi\alpha) - 1)\exp(im\varphi) \tag{4.10}$$

The trick is now to get rid of the sign(m + α) by writing the scattering amplitude as two sums, where one sum is the complex conjugate of the original sum.

$$\lim_{\varepsilon\to 0}\frac{\exp(-i\alpha\varphi)}{-(2\pi k)^{\frac{1}{2}}}\left(\sum_{m=0}^{\infty}(\exp(-i\pi\alpha) - 1)\exp(im\varphi - m\varepsilon) + \sum_{m=1}^{\infty}(\exp(i\pi\alpha) - 1)\exp(-im\varphi - m\varepsilon)\right) \tag{4.11}$$

By evaluating the geometric series, we finally obtain the scattering amplitude.

$$f(\varphi) = \left(\frac{2\pi}{k}\right)^{\frac{1}{2}} \left(-\operatorname{sign}(m+\alpha)\frac{\pi\alpha}{2}(\cos(\pi\alpha)-1) + i\frac{\sin(\pi\alpha)}{\pi}\frac{\exp(-i\alpha\varphi)}{\exp(-i\varphi)+1}\right) \tag{4.12}$$

The differential cross section for any angle $\varphi \neq 0$, with k = 1, becomes:

$$\frac{d}{d\varphi}\sigma = \frac{\sin^2(\pi\alpha)}{2\pi\cos^2\left(\frac{\varphi}{2}\right)} \tag{4.13}$$

We have, therefore, verified that the interference pattern predicted by means of the path integral method is the same as the one that was obtained by Aharonov and Bohm who used the method of the Schrodinger equation.

5 Conclusion

5.1 Discussion

We have looked at the AB-effect and followed the steps to determine the associated wave function using the path integral method. Although some aspects of this method raised doubt on its integrity, the final result for the interference pattern was shown to be equal to the predicted effect using the method of solving the Schrodinger equation. The path integral method allowed for a more intuitive derivation of the AB-effect. Even though there were some problems with the definition of the Hamiltonian and subsequently the time evolution of the system, these problems were solved in a very natural way by means of summing over all homotopy classes. An interesting question that remains is to what extent the higher order homotopy classes add to the full propagator, i.e. do particles really loop around a particular point is space multiple times?

The Hamiltonian for the AB-setup was not self-adjoint due to the fact that the space we were dealing with was multiply connected. A family of extensions was found that uniquely defined the dynamics of the system. This was done by looking for an extra piece of information. The information resided in the boundary condition and the quantity of interest was the magnetic flux F of the solenoid. Using this extra parameter, the wave function could be found and the AB-effect could be explained. From this point of view, the precise details of the magnetic vector potential A were completely ignored. Aharonov and Bohm themselves suggested that the vector potential directly influences the wave function, but the potential is not necessarily needed to do the mathematics of this particular problem.

The insight by Aharonov and Bohm showed an aspect of quantum mechanics that to this day remains controversial and is sometimes misunderstood. The seemingly nonlocal interaction of the magnetic flux with the particles is surprising and raises questions about the true nature of the electric scalar potential and the magnetic vector potential. It would be interesting to find proof of the same effect in the other three fundamental forces of nature.

Bibliography

Aharonov, Y. and D. Bohm. “Further Considerations on Electromagnetic Potentials in the Quantum Theory.” Physical Review (1961): 1511-1524.

—. “Significance of Electromagnetic Potentials in the Quantum Theory.” The Physical Review (1959): 485-491.

Bernido, C. C. and A. Inomata. “Path Integrals with a Periodic Constraint: The Aharonov-Bohm Effect.” Journal of Mathematical Physics (1981): 715-718.

Berry, M. V. “Quantal Phase Factors accompanying Adiabatic Changes.” Proceedings of the Royal Society A (1984): 45-57.

—. “The Geometric Phase.” Scientific American (1988): 46-52.

Bonneau, G., J. Faraut and G. Valent. “Self-Adjoint Extensions of Operators and the Teaching of Quantum Mechanics.” American Journal of Physics (2001): 322-331.

Boyer, T. H. “Does the Aharonov-Bohm Effect Exist?” Foundations of Physics (2000): 893-905.

Carlen, E. A. and M. I. Loffredo. “The Correspondence Between Stochastic Mechanics and Quantum Mechanics on Multiply Connected Configuration Spaces.” Physics Letters A (1989): 9-13.

Chambers, R. G. “Shift of an Electron Interference Pattern by Enclosed Magnetic Flux.” Physical Review Letters (1960): 3-5.

Dowker, J. S. “Quantum Mechanics and Field Theory on Multiply Connected and on Homogeneous Spaces.” Journal of Physics A (1972): 936-943.

Garbaczewski, P. and W. Karwowski. “Impenetrable Barriers and Canonical Quantization.” American Journal of Physics (2004): 924-933.

Gerry, Christopher C. and Vijay A. Singh. “Feynman Path-integral Approach To The Aharonov-Bohm Effect.” Physical Review D (1979): 2550-2554.

Inomata, A. and V. A. Singh. “Path Integrals with a Periodic Constraint: Entangled Strings.” Journal of Mathematical Physics (1978): 2318-2323.

Laidlaw, M. G. G. and C. Morette-DeWitt. “Feynman Functional Integrals for Systems of Indistinguishable Particles.” Physical Review D (1971): 1375-1378.

Le Bellac, M. Quantum Physics. Cambridge: Cambridge University Press, 2006.

Peak, D. and A. Inomata. “Summation over Feynman Histories in Polar Coordinates.” Journal of Mathematical Physics (1969): 1422-1428.

Sakoda, S. and M. Omote. “Difference in the Aharonov-Bohm Effect on Scattering States and Bound States.” Hawkes, P. W. Advances in Imaging and Electron Physics. San Diego: Academic Press, 1999. 101-171.

Schulman, L. S. “A Path Integral for Spin.” Physical Review (1968): 1558-1569.

—. “Approximate Topologies.” Journal of Mathematical Physics (1971): 304-308.

—. Techniques And Applications Of Path Integration. New York: John Wiley & Sons, Inc., 1981.

Shankar, R. Principles of Quantum Mechanics. New York: Plenum Press, 1994.

Thaller, B. Visual Quantum Mechanics. New York: Springer-Verlag New York, Inc., 2000.

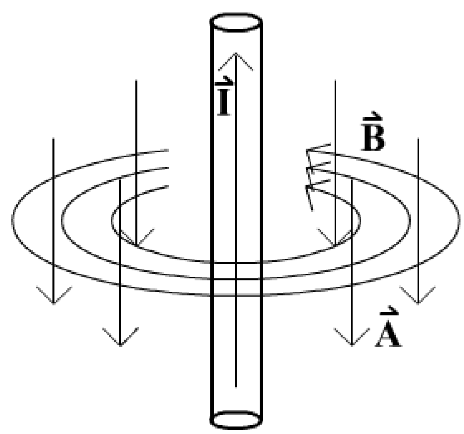
Appendix

The Magnetic Vector Potential

In this paper, we have used the magnetic vector potential A. Here, we shall derive the expression for A for a long, straight, current-carrying conductor or wire. We will then build on this to find the vector potential for a solenoid. In each case, an idealisation is used where the wire and the solenoid stretch from minus infinity to plus infinity. Furthermore, the radii of the wire and of the solenoid are taken to be infinitesimal, i.e. R is limited to 0 while keeping the total flux constant.

Wire

In this section, we will derive the magnetic vector potential for an infinitely long wire. This will be done by examining the current flowing through the wire and by determining what kind of magnetic field this produces.



The wire is placed along the z-axis as shown in figure 2. It conducts a charge density ρ(c) with velocity v pointing in the positive z-direction.

$$\vec{J} = \rho_c \vec{v} = J_0 \hat{z} \tag{A.01}$$

$$\text{where } J_0 = \rho_c v_0 \tag{A.02}$$

The current running through the wire is the total current density over the cross section of the wire, which is assumed to remain constant even when the limit of R(w) → 0 is taken.

$$\vec{I} = \pi {R_w}^2 \vec{J} = I_0 \hat{z} \tag{A.03}$$

$$\text{where } I_0 = \pi {R_w}^2 J_0 \tag{A.04}$$

Ampere's circuital law states that the circuit integral of the magnetic field over a loop is proportional to the electric flux piercing through the surface enclosed by the loop. In this case, any circular loop with circumference 2πρ, enclosing the wire, runs over the angular component of the magnetic field and is proportional to the current running through the wire.

$$\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_0 \tag{A.05}$$

$$2\pi\rho B_\varphi = \frac{4\pi}{c} I_0 \tag{A.06}$$

An expression for the angular component of B can, then, easily be found. It follows from the symmetry of this setup that the magnetic field is independent of z and of φ; B only has an angular component, which falls off with 1/ρ.

$$B_\varphi = \frac{2}{c\rho} I_0 = B_0 \frac{1}{\rho} \tag{A.07}$$

$$\text{where } B_0 = \frac{2}{c} I_0 \tag{A.08}$$

$$\vec{B} = B_\varphi \hat{\varphi} = B_0 \frac{1}{\rho} \hat{\varphi} \tag{A.09}$$

The magnetic potential is defined as that of which the curl is B. Again, the symmetry of the problem helps us to determine that A only has a component in the z-direction.

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial}{\partial\rho} A_z \hat{\varphi} \tag{A.10}$$

An expression for the z component of A, and then also for the vector potential itself, can be found. A seems to go with the log of ρ and points in the negative z-direction, i.e. antiparallel to the current.

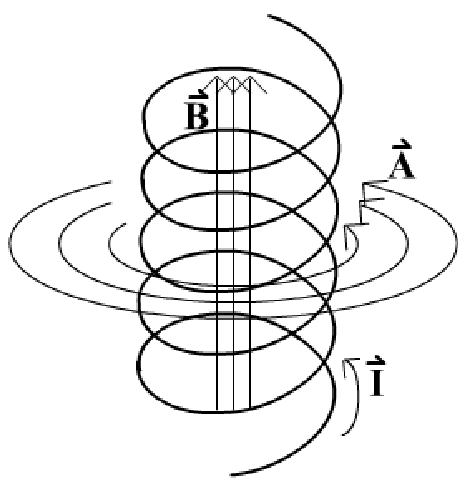
$$A_z = -B_0 \log\rho = -A_0 \log\rho \tag{A.11}$$

$$\text{where } A_0 = B_0 \tag{A.12}$$

$$\vec{A} = A_z \hat{z} = -A_0 \log\rho\, \hat{z} \tag{A.13}$$

Solenoid

The setup of the solenoid is quite similar to that of the wire. In this case, the wire is circled round with a radius of R(s). This will produce a magnetic field inside the solenoid, pointing in the positive z-direction as shown in figure 3.



Although magnetic field lines always loop back onto themselves, there is no magnetic field outside of the solenoid because it is infinitely long, i.e. the total returning magnetic flux outside the solenoid is spread out over an infinitely large surface and the flux at any point outside the solenoid is, therefore, 0. The total current running through the wire of the solenoid is still I(0), but now points in the φ-direction.

$$\vec{I} = \pi {R_w}^2 \vec{J} = I_0 \hat{\varphi} \tag{A.14}$$

$$\text{where } I_0 = \pi {R_w}^2 J_0 \tag{A.15}$$

Again, Ampere's circuital law can be used to determine the strength of the magnetic field.

$$\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_0 \tag{A.16}$$

A loop can be drawn through the solenoid which encloses n cross sections of the wire and stretches over length l in the z-direction through the magnetic field. Because the magnetic field points only in the z-direction, the parts of the loop perpendicular to it amount to 0. Furthermore, the magnetic field outside the solenoid is also 0 and the part which loops back will feel no magnetic field.

$$B_z = \frac{n}{l} \frac{4\pi}{c} I_0 = B_0 \tag{A.17}$$

$$\vec{B} = B_z \hat{z} = B_0 \hat{z} \tag{A.18}$$

$$\text{for } \rho \leq R_s \tag{A.19}$$

The vector potential for the solenoid can be found by realising that the magnetic field is the curl of A. The symmetry of the problem suggests that A cannot depend on φ or z.

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho A_\rho \right) \hat{z} \tag{A.20}$$

Furthermore, Stokes's theorem relates the surface integral of a curl of some vector field to the circuit integral around a loop through the vector field. As such, the curl of the magnetic vector potential A, which is equal to the magnetic field B, can be integrated over the surface of the solenoid. This is simply the magnetic flux piercing the surface, and is equal to the circuit integral over a loop through A.

$$\int_S \left(\vec{\nabla} \times \vec{A} \right) da = \int_S B_0 \, da = \oint \vec{A} \cdot d\vec{l} \tag{A.21}$$

$$\pi {R_s}^2 B_0 = F = 2\pi \rho A_\varphi \tag{A.22}$$

The only component of A that is nonzero is the angular component, which follows from the symmetry of the problem. Therefore, an expression for A can easily be obtained.

$$A_\varphi = \frac{\pi {R_s}^2 B_0}{2\pi} \frac{1}{\rho} = A_0 \frac{1}{\rho} \tag{A.23}$$

$$\text{where } A_0 = \frac{1}{2} {R_s}^2 B_0 \tag{A.24}$$

$$\vec{A} = A_\varphi \hat{\varphi} = A_0 \frac{1}{\rho} \hat{\varphi} \tag{A.25}$$

The magnetic vector potential A for a solenoid circles around the solenoid, pointing in the φ-direction with a magnitude which is determined by the magnetic flux within the solenoid. As we already mentioned, there is no magnetic field outside of the solenoid in this idealisation, but there is a vector potential which contained information about the magnetic field.

The Exact Solution

The Schrodinger Equation

Before we begin to derive the Schrodinger equation for the AB-setup, we must establish some basic relations.

The Hamiltonian acting on a wave function:

$$H\psi = E\psi \tag{A.26}$$

The Hamiltonian:

$$H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 \tag{A.27}$$

The energy:

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{A.28}$$

$$\text{where } p = \hbar k \tag{A.29}$$

When we realise, with foresight, that the general form of ψ is a plane wave, we find:

$$\psi = \exp(ikx + i\omega t) \tag{A.30}$$

$$\nabla\psi = ik\psi = \frac{i}{\hbar}p\psi \tag{A.31}$$

$$p = -i\hbar\nabla \tag{A.32}$$

Therefore, the Schrodinger equation becomes:

$$\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2 \psi = \frac{\hbar^2}{2m}k^2\psi \tag{A.33}$$

$$\left(\vec{\nabla} - i\frac{q}{c\hbar}\vec{A}\right)^2 \psi = -k^2\psi \tag{A.34}$$

$$\left(\vec{\nabla}^2 - i\frac{q}{c\hbar}\left(\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\right) - \frac{q^2}{c^2\hbar^2}\vec{A}^2 + k^2\right)\psi = 0 \tag{A.35}$$

In this system, the magnetic vector potential, under the correct gauge, is the one obtained for a solenoid.

$$\vec{A} = \frac{A_0}{\rho}\hat{\varphi} \tag{A.36}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\varphi}\frac{A_0}{\rho} = \frac{A_0}{\rho^2}\frac{\partial}{\partial\varphi} \tag{A.37}$$

$$\vec{A}\cdot\vec{\nabla} = \frac{A_0}{\rho}\frac{1}{\rho}\frac{\partial}{\partial\varphi} = \frac{A_0}{\rho^2}\frac{\partial}{\partial\varphi} \tag{A.38}$$

When we substitute these relations in, we obtain:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} - 2i\frac{q}{c\hbar}\frac{A_0}{\rho^2}\frac{\partial}{\partial\varphi} - \frac{q^2}{c^2\hbar^2}\frac{{A_0}^2}{\rho^2} + k^2\right)\psi = 0 \tag{A.39}$$

$$\text{with } \vec{\nabla}^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} \tag{A.40}$$

We can simplify the Schrodinger equation by substituting in α.

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+2\mathrm{i}\frac{\alpha}{\rho^2}\frac{\partial}{\partial\varphi}-\frac{\alpha^2}{\rho^2}+\mathrm{k}^2\right)\psi=0 \tag{A.41}$$

$$\text{where } \alpha=-\frac{\mathrm{q}}{\mathrm{c}\hbar}\mathrm{A}_0 \tag{A.42}$$

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi}+\mathrm{i}\alpha\right)^2+\mathrm{k}^2\right)\psi=0 \tag{A.43}$$

The Bessel Functions

The Schrodinger equation for the AB-setup looks very similar to the Bessel equation.

$$\mathrm{x}^2\frac{\partial^2}{\partial\mathrm{x}^2}\mathrm{J}_\nu+\mathrm{x}\frac{\partial}{\partial\mathrm{x}}\mathrm{J}_\nu+(\mathrm{x}^2-\nu^2)\mathrm{J}_\nu=0 \tag{A.44}$$

When the following substitution is made, we get one step closer to our Schrodinger equation.

$$\rho^2\frac{\partial^2}{\partial\rho^2}\mathrm{J}_\nu+\rho\frac{\partial}{\partial\rho}\mathrm{J}_\nu+(\mathrm{k}^2\rho^2-\nu^2)\mathrm{J}_\nu=0 \tag{A.45}$$

$$\text{with } \mathrm{x}\;\rightarrow\;\rho\mathrm{k} \tag{A.46}$$

We divide both sides by ρ^2.

$$\frac{\partial^2}{\partial\rho^2}\mathrm{J}_\nu+\frac{1}{\rho}\frac{\partial}{\partial\rho}\mathrm{J}_\nu+\frac{1}{\rho^2}(-\nu^2)\mathrm{J}_\nu+\mathrm{k}^2\mathrm{J}_\nu=0 \tag{A.47}$$

We see that we can determine ν for our setup.

$$-\nu^2=\left(\frac{\partial}{\partial\varphi}+\mathrm{i}\alpha\right)^2 \tag{A.48}$$

$$\nu=-\mathrm{i}\frac{\partial}{\partial\varphi}+\alpha=\mathrm{m}+\alpha \tag{A.49}$$

$$\text{where } \mathrm{m}=-\mathrm{i}\frac{\partial}{\partial\varphi} \tag{A.50}$$

Therefore, the solutions to the Schrodinger equation for the AB-setup are Bessel functions where the argument ν is replaced by m + α.

The Cross Section and Scattering Amplitude

Here, we will derive the expression for the differential cross section dσ/dφ in terms of the scattering amplitude f(φ) (Shankar; Le Bellac). We take an arbitrary wave function that is scattered due to some potential at the origin. The strength of this scattered wave falls off with the square root of 1/ρ because it spreads as circular waves from the origin to infinity in all directions, the square root coming from the fact that we are only dealing with the probability amplitude.

$$\psi=\psi_\mathrm{i}+\psi_\mathrm{s}=\exp(-\mathrm{i}\rho\cos(\varphi))+\mathrm{f}(\varphi)\frac{\exp(\mathrm{i}\rho)}{(\rho)^{\frac{1}{2}}} \tag{A.51}$$

The number of particles, or better, the probability N(s) that scatters into the infinitesimal angle dφ per unit time is the probability current N(i) of the incident wave crossing a unit surface multiplied by the infinitesimal ratio dσ for that particular angle.

$$dN_s = N_i \frac{d}{d\varphi}\sigma d\varphi \tag{A.52}$$

To find the differential cross section, we need to know the probability that is flowing into dφ relative to the incident probability current density. Let us consider the probability current density for the scattering part of the wave function first. We know that we can separate the incident part and the scattering part when we observe the wave function at an angle φ other than φ = 0, the angle at which the incident wave is propagating. We must also take care that we do not include the interference terms. The interference terms between the incident wave and the scattering wave disappear when we observe at ρ → ∞.

$$\vec{J_s} = \frac{\hbar}{2mi}\left(\psi_s{}^*\vec{\nabla}\psi_s - \psi_s\vec{\nabla}\psi_s{}^*\right) \tag{A.53}$$

$$\text{with } \vec{\nabla} = \frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\hat{\varphi} \tag{A.54}$$

The second term of the del operator can be neglected for ρ → ∞.

$$\frac{\partial}{\partial\rho} f(\varphi)\frac{\exp(i\rho)}{(\rho)^{\frac{1}{2}}}\hat{\rho} = f(\varphi)\, i\frac{\exp(i\rho)}{(\rho)^{\frac{1}{2}}}\hat{\rho} + O\left(\rho^{-\frac{2}{3}}\right) \tag{A.55}$$

Therefore, the current density becomes:

$$\vec{J_s} = \frac{\hbar}{m}\frac{|f(\varphi)|^2}{\rho}\hat{\rho} \tag{A.56}$$

The probability N(s) that scatters into the infinitesimal angle dφ per unit time is equal to the integral of the current density j(s) over the line dl stretched by the angle dφ. At a distance ρ, this line has length ρdφ.

$$N_s = \int d\varphi\, \rho\vec{J_s} \cdot \hat{\rho} \tag{A.57}$$

$$dN_s = \frac{\hbar}{m}|f(\varphi)|^2 d\varphi \tag{A.58}$$

From the expression above for the infinitesimal number of particles dN(s) it follows that:

$$\frac{d}{d\varphi}\sigma = |f(\varphi)|^2 \tag{A.59}$$

The calculation of the differential cross section, then, becomes trivial.